SELF-SIMILAR SOLUTIONS OF THE LAMINAR BOUNDARY LAYER EQUATIONS FOR A COMPRESSIBLE FLUID INCLUDING HEAT TRANSFER

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PMM Vol.22, No.2, 1958, pp.274-279

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(Received 12 January 1958)

In order to obtain useful solutions to various problems in boundary layer theory, attempts have been made to transform the partial differential equations to ordinary ones. Thus the so-called similar or self-similar solutions have been obtained. For the case of flow of an incompressible fluid without heat transfer the question of describing all self-similar solutions has been studied conclusively [1, 2].

In the case of a compressible fluid the results of various studies to determine self-similar solutions (in the sense of the definition given below) for flow past a plate were described in [3]. Certain self-similar solutions were found for a compressible boundary layer without heat transfer in [4], and including heat transfer in [5]. However, [5] did not exhaust all the self-similar solutions in the sense indicated. The present work enumerates, for the case of a compressible fluid including heat transfer, all self-similar solutions of the boundary layer equations. It is shown that no other self-similar solutions exist.

The motion of a compressible fluid in the boundary layer is determined by the following system of equations [3]

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\qquad \frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$
$$\rho\left(u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}\right)=\frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu\frac{\partial\theta}{\partial y}\right)+\left(1-\frac{1}{\sigma}\right)\frac{\partial}{\partial y}\left(\frac{\mu}{Ec_p}u\frac{\partial u}{\partial y}\right)\qquad(1)$$

where

$$\theta=T+\frac{u^2}{2Ec_p},\qquad \sigma=\frac{\mu c_p}{\varkappa}$$

Here p, ρ, T and μ are the pressure, density, temperature and coefficient of viscosity, u and v are the velocity components along the x and y axes, θ is the stagnation temperature of the stream, E the mechanical equivalent of heat, c_p the specific heat at constant pressure, κ the

Reprint Order No. PMM 34.

coefficient of thermal conductivity and σ the Prandtl number. Furthermore p_0, ρ_0 and T_0 denote the stagnation pressure, density and temperature of the external stream (that is, for $y \to \infty$), U the speed of the external stream and μ_0 the value of the coefficient of viscosity at $T = T_0$. Henceforth it is assumed that $\mu/\mu_0 = T/T_0$.

The value of the temperature at the wall is denoted by $\tau(x)$. Then the boundary conditions for the system (1) will have the form

$$u = v = 0, \qquad \theta = \tau(x) \qquad \text{at} \quad y = 0;$$
$$u = U(x), \qquad \theta = T_0 \qquad \text{at} \quad y = \infty$$

Similar to what was done in [4], we introduce new variables analogous to those of Dorodnitsyn:

$$\xi = \int_0^x \left(\frac{p}{p_0}\right)^\alpha dx, \qquad \eta = \int_0^y \frac{\rho}{\rho_0}\, dy$$

where α is a certain constant, the choice of which will be discussed later. We introduce the symbols

$$U_{\max} = \frac{2k}{k-1}\, gRT_0, \qquad V = u\left(\frac{p_0}{p}\right)^\alpha \frac{\partial\eta}{\partial x} + v\left(\frac{p_0}{p}\right)^{\alpha-1} \frac{T_0}{T}$$

and dimensionless quantities according to the formula

$$U = \bar{U} U_{\max}, \quad u = \bar{u} U_{\max}, \quad V = \bar{V}\frac{\nu_0}{L}, \quad \xi = \bar{\xi}\,\frac{U_{\max} L^2}{\nu_0}$$
$$\eta = \bar{\eta} L, \qquad \theta = (\bar{\theta} + 1)\, T_0, \qquad \tau = (\bar{\tau}_0 + 1)\, T_0$$

where L is a certain characteristic length, k the adiabatic exponent and R the gas constant. We transform the system of equations (1) to the form

$$u\frac{\partial u}{\partial \xi} + V\frac{\partial u}{\partial \eta} = (\theta - u^2)\frac{UU'}{1-U^2} + (1-U^2)^{-\delta}\frac{\partial^2 u}{\partial \eta^2} \tag{2}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0, \qquad \delta = \frac{k(\alpha-1)}{k-1} \tag{3}$$

$$u\frac{\partial \theta}{\partial \xi} + V\frac{\partial \theta}{\partial \eta} = (1-U^2)^{-\delta}\left\{\frac{1}{\sigma}\frac{\partial^2\theta}{\partial\eta^2} + 2\left(1-\frac{1}{\sigma}\right)\frac{\partial}{\partial\eta}\left(u\frac{\partial u}{\partial\eta}\right)\right\} \tag{4}$$

Here the bars have been dropped from all quantities. All quantities considered henceforth are dimensionless. From this system three quantities are to be determined subject to the following boundary conditions:

$$u = v = 0, \qquad \theta = \tau(\xi) \qquad \text{at} \quad \eta = 0$$
$$u = U(\xi), \qquad \theta = 0 \qquad \text{at} \quad \eta = \infty$$

Let ξ and ξ^* be two arbitrary fixed points on the wall in the ξ,η plane. Also $u(\xi,\eta)$, $\theta(\xi,\eta)$ and $u(\xi^*,\eta)$, $\theta(\xi^*,\eta)$ are the corresponding profiles of the quantities u and θ at those points. The question arises whether it is possible to perform a similarity transformation of the η, u, and θ axes (with, generally speaking, different coefficients of expansion or contraction) such that the profiles $u(\xi,\eta)$ and $\theta(\xi,\eta)$ become precisely congruent to the profiles $u(\xi^*,\eta)$ and $\theta(\xi^*,\eta)$. If this is possible, then

$$u(\xi,\eta) = Au(\xi^*, B\eta), \qquad \theta(\xi,\eta) = C\theta(\xi^* B\eta)$$

It is understood that the factors A, B, C depend on ξ for a fixed ξ^*.

Definition: If the indicated transformation is possible for arbitrary ξ, then we say that the problem has a self-similar solution.

Since ξ^* is fixed, taking $X = B(\xi)\eta$ and introducing instead of $f(X)$ the function

$$\varphi(X) = \int f(X)\,dX$$

we obtain

$$u(\xi,\eta) = A(\xi) f(X), \qquad \theta(\xi,\eta) = C(\xi) g(X) \tag{5}$$

The question is now investigated for which distributions of velocity $U(\xi)$ and wall temperature $\tau(\xi)$ the system (2) - (4) admits of solutions of the form (5).

If $\eta = 0$ then $X = 0$; if $\eta \to \infty$ then $X \to \infty$. Therefore from the boundary conditions and the first of the relations (5) we find $\phi'(0) = 0$. Moreover $U(\xi) = A(\xi)\,\phi'(\infty)$. There is no loss of generality in normalizing $\phi'(X)$ such that $\phi'(\infty) = 1$. Then $A(\xi) = U(\xi)$. Analogously, from the second of the relations (5) and the boundary conditions we obtain $g(\infty) = 0$. Also $\tau(\xi) = C(\xi)\ g(0)$. Normalizing $g(X)$ such that $g(0) = 1$, we obtain $C(\xi) = \tau(\xi)$. Now the relations (5) assume the form

$$u(\xi,\eta) = U(\xi)\varphi'(X), \qquad \theta(\xi,\eta) = \tau(\xi) g(X) \tag{6}$$

From the first of the relations (6) and equation (3) we obtain

$$V(\xi,\eta) = \left\{\frac{UB'}{B^2} - \frac{U'}{B}\right\}\varphi(X) - \frac{UB'}{B^2} X\varphi'(X) \tag{7}$$

Since $V(\xi,0) = 0$ we conclude at once that $\phi(0) = 0$.

Substituting relations (6) and (7) into equations (2) and (4), we obtain after some transformations

$$(1-U^2)\left(\frac{UB'}{B} - U'\right)\varphi\varphi'' + U'\varphi'^2 = \tau U'g + \frac{B^2(1-U^2)}{(1-U^2)^\delta}\varphi'''$$

$$\left(\frac{UB'}{B} - U'\right)\varphi g' + U\frac{\tau'}{\tau}g\varphi' = (1-U^2)^{-\delta}\left\{\frac{1}{\sigma}B^2 g'' + \left(1-\frac{1}{\sigma}\right)\frac{B^2U^2}{\tau}(\varphi'^2)''\right\} \tag{8}$$

In these equations ϕ and g are unknown functions of X satisfying the boundary conditions

$$\varphi(0) = \varphi'(0) = 0, \qquad \varphi'(\infty) = 1, \qquad g(0) = 1, \qquad g(\infty) = 0$$

and U, B, and τ are functions of ξ. If similar solutions exist, this means that for proper choice of the function $B(\xi)$ equation (8) becomes an ordinary differential equation.

In the present work all functions U and τ are found for which such a choice of $B(\xi)$ is possible. The cases $\sigma \neq 1$ and $\sigma = 1$ are investigated separately. In the first case the following theorem arises:

Theorem 1. If $\sigma \neq 1$, equation (8) reduces to an ordinary differential equation only if simultaneously $U =$ const and τ = const. Then the function $B(\xi)$ is determined uniquely to within a constant multiplier.

Proof. We set $a = 1$ in our equations; that is, $\delta = 0$. If $U =$ const then equations (8) take the form

$$\varphi''' - U\frac{B'}{B^3}\varphi\varphi'' = 0$$
$$U\frac{B'}{B^3}\varphi g' + \frac{U}{B^2}\frac{\tau'}{\tau}g\varphi' = \frac{1}{\sigma}g'' + \left(1 - \frac{1}{\sigma}\right)\frac{U^2}{\tau}(\varphi'^2)'' \tag{9}$$

In order that the first be an ordinary differential equation, it is necessary that the expression UB'/B^3 should not depend on ξ; that is UB'/B^3 = const. Furthermore, all coefficients in the second equation must be constants. This means that U^2/τ = const and consequently τ = const and $\tau' = 0$. We now show how to find the function $B(\xi)$. By multiplying it by a constant factor we can change the quantity UB'/B^3. Thus we will choose $B(\xi)$ such that $UB'/B^3 = -2$. Hence it is easy to obtain

$$B = \frac{1}{\sqrt{a\xi + b}} \qquad \left(a = \frac{4}{U}\right)$$

The system of equations under consideration takes the form

$$\varphi''' + 2\varphi\varphi'' = 0, \quad g'' + 2\sigma\varphi g' = (1 - \sigma)\frac{U^2}{\tau}(\varphi'^2)''$$

This system was considered in detail in [3].

It remains to show that equation (8) does not reduce to an ordinary differential equation of $U \neq$ const. Now in this case dividing the first of equations (8) by u' we obtain

$$(1 - U^2)\left(\frac{UB'}{U'B} - 1\right)\varphi\varphi'' + \varphi'^2 = \tau g + \frac{B^2(1 - U^2)}{U'}\varphi''' \tag{9}$$

This is an ordinary differential equation if

$$\tau = \text{const}, \qquad (1 - U^2)\left(\frac{UB'}{U'B} - 1\right) = \text{const}, \qquad \frac{B^2(1 - U^2)}{U'} = \text{const} \tag{10}$$

After multiplication of the second of equations (8) by $(1 - U^2)/U'$ we find

$$(1 - U^2)\left(\frac{UB'}{U'B} - 1\right)\varphi g' = \frac{B^2(1 - U^2)}{U'}\,\frac{1}{\sigma}g'' + \left(1 - \frac{1}{\sigma}\right)\frac{B^2U^2(1 - U^2)}{U'}(\varphi'^2)''$$

Hence we conclude that

$$\frac{B^2U^2(1 - U^2)}{U'} = \text{const} \tag{10}$$

Comparing with (10a) we obtain U^2 = const, which contradicts the assumption. Thus the theorem is proved.

Now we consider the case when $\sigma = 1$. In the proof of Theorem 1 it was shown that under the condition $U \neq 0$ in order for the first of equations (8) and hence of (9a) to reduce to an ordinary differential equation it was necessary that τ = const.

1. This means that in order for the system (8) to be a system of ordinary differential equations it is necessary that one of the conditions U = const or τ = const be satisfied.

We consider first the case when U = const. Assuming again $\alpha = 1$, $\delta = 0$ and $\sigma = 1$ we write the system (8) as

$$\varphi''' - 2\varphi\varphi'' = 0$$

$$2\varphi g' - \frac{U}{B^2}\,\frac{\tau'}{\tau}\,g\varphi' = g'' \qquad \left(B^2 = \frac{1}{a\xi + b}, \quad a = \frac{4}{U}\right)$$

Hence it follows that

$$\frac{U}{B^2}\,\frac{\tau'}{\tau} = (4\xi + Ub)\,\frac{\tau'}{\tau} = \text{const}$$

Setting $\frac{1}{4} Ub = c$ and $(4\xi + Ub)\tau'/\tau = 4n$, we find that to within a constant multiplier $\tau = (\xi + c)^n$ and the system of equations assumes the form

$$\varphi''' + 2\varphi\varphi'' = 0, \qquad g'' + 2\varphi g' + 4n\varphi' g = 0 \tag{11}$$

The first of these equations is the well-known Blasius equation, whose solution is tabulated. The second is linear in g and can be easily solved with sufficient accuracy, for example by Galerkin's method. We have proved the proposition.

2. If $\sigma = 1$ and U = const, equations (8) reduce to ordinary ones only when $\tau = (\xi + c)^n$. Then the system assumes the form (11).

Remark. If $\sigma = 1$ and U = const, equations (8) reduce to ordinary ones also for $\tau = Ae^{b\xi}$. However, it is then necessary that B = const. and the first of equations (8) assumes the form $\phi''' = 0$. But the solutions of this equation do not satisfy the boundary conditions of the problem.

We now consider the latter case: $\sigma = 1$, τ = const. Then the system (8) assumes the form

$$(1 - U^2)\left(\frac{UB'}{U'B} - 1\right)\varphi\varphi'' + \varphi'^2 = \tau g + \frac{B^2(1 - U^2)}{U'(1 - U^2)^\delta}\,\varphi'''$$

$$(1 - U^2)\left(\frac{UB'}{U'B} - 1\right)\varphi g' = \frac{B^2(1 - U^2)}{U'(1 - U^2)^\delta}\,g'' \tag{12}$$

We investigate for what $U(\xi)$ these equations become ordinary ones. This obviously occurs only when simultaneously

$$(1 - U^2)\left(\frac{UB'}{U'B} - 1\right) = \text{const}, \qquad \frac{B^2(1 - U^2)}{U'(1 - U^2)^\delta} = \text{const} \tag{13}$$

Differentiating the second relation, we readily find

$$\frac{B'}{B}=\frac{1}{2}\left\{\frac{U''}{U'}-\frac{2(\delta-1)UU'}{1-U^2}\right\}$$

We substitute this expression into the first of equations (13), also denoting the constant on the right side by β^{-1} Then

$$\frac{1}{2}\frac{UU''}{U'^2}=1-\frac{\beta^{-1}-(\delta-1)U^2}{1-U^2} \tag{14}$$

We recall that $\delta = k(a-1)/(k-1)$ and that a can be chosen arbitrarily. We first choose a so that $\delta - 1 = \beta^{-1}$. In this case the equation acquires the form: $UU'' = 2(1-\beta^{-1})U'^2$. It is easily seen that all solutions of this equation are expressed as powers or exponentials as follows:

$$U=(a\xi+b)^n \quad \text{for } \beta\neq 2, \qquad U=ae^{b\xi} \quad \text{for } \beta=2 \tag{15}$$

Here also in the first case $\beta = 2n/(n+1)$. Now $B(\xi)$ can be found from the second of the equations (13). Without loss of generality H can be normalized such that

$$\left|\frac{B^2(1-U^2)}{U'(1-U^2)^\delta}\right|=\left|\frac{1}{\beta}\right| \quad \text{for} \quad \frac{B^2(1-U^2)}{U'(1-U^2)^\delta}=\pm\frac{1}{\beta}$$

Here the sign on the right side agrees with the corresponding sign in the relation: sign $U' = \pm$ sign β. In the case of a plus sign the system (12) takes the form

$$\varphi'''+\varphi\varphi''=\beta(\varphi'^2-\tau g), \qquad g''+\varphi g'=0 \qquad \left(B=\sqrt{\frac{U'}{\beta}(1-U^2)^{1/\beta}}\right)$$

and in the case of a minus sign

$$-\varphi'''+\varphi\varphi''=\beta(\varphi'^2-\tau g), \qquad g''-\varphi g'=0 \qquad \left(B=\sqrt{-\frac{U'}{\beta}(1-U^2)^{1/\beta}}\right)$$

Now if a is chosen such that $\delta - 1 = \beta^{-1}$, then in the ξ,η plane (where the variable ξ depends on a) equations (8) can be reduced to ordinary ones if $U(\xi)$ is expressed in the form (15). However, if a is chosen otherwise, in order to find those $U(\xi)$ for which equations (8) become ordinary ones it is necessary to solve equation (14). Recuding its order we obtain

$$\frac{dU}{d\xi}=K\frac{U^\gamma}{(1-U^2)^{\frac{\gamma}{2}+\delta-2}} \qquad \left(2-\frac{2}{\beta}=\gamma\right) \tag{16}$$

where K is some constant. Hence it is possible to find the relation between U and ξ simply by integration. We notice, among other things, that for $\delta - 1 = \beta^{-1}$ this equation takes the form $dU/d\xi = KU^\gamma$, all solutions of which are given by equation (15).

If a is chosen such that $\delta - 1 = \beta^{-1}$ then the rule of variation of $U(\xi)$ for which self-similar solutions are possible is expressed most simply in the appropriate ξ,η plane. But if a is chosen otherwise, then the variable ξ is different and the law for $U(\xi)$ changes. Nevertheless, wahtever a may be, upon returning to the variables x, y we find always the same rule for $U(x)$. What is indicated above permits formulation of the proposition:

3. If $\sigma = 1$ and r = const, then equations (8) reduce to ordinary ones only when the law of distribution of the velocity $U(\xi)$ is expressed, for suitable choice of a, in the form (15).

Remark. In investigating the system (12) we put

$$(1 - U^2)\left(\frac{UB'}{U'B} - 1\right) = \frac{1}{\beta}$$

assuming that this quantity was different from zero. However, if it is set equal to zero the second of equations (12) immediately gives $g'' = 0$. The solution of this equation is a linear function, which cannot satisfy the boundary conditions for $g(X)$. So in this case a self-similar solution clearly does not exist. We note that in the absence of heat transfer the assumption that $UB'/U'B - 1 = 0$ corresponds to a contracting duct in the ξ,η plane and leads to a self-similar solution [4].

The proofs of statements 1, 2 and 3 can be combined into the following theorem.

Theorem 2. In order that equations (8) be ordinary differential equations when $\sigma = 1$, it is necessary and sufficient that one of the following conditions be realized:

$$1.\ U = \text{const},\quad \tau = A(\xi + c)^n \qquad 2.\ \tau = \text{const},\quad U = (a\xi + b)^n$$
$$3.\ \tau = \text{const},\quad U = ae^{b\xi}$$

Here the function $B(\xi)$ is determined uniquely to within a multiplicative constant.

For these systems it is still necessary to solve each time a one-dimensional boundary value problem on the half axis. In the present work the questions have not been considered of existence and uniqueness of the solutions of the indicated boundary value problems. These problems are very complicated and have not been investigated very far. In [7], for example, it is shown that in the case of an incompressible fluid without heat transfer (that is, $rg = 1$) the equation $\phi''' + \phi\phi'' = \beta(\phi'^2 - 1)$ for $\beta < 0$ has infinitely many solutions satisfying all the boundary conditions. In this case from all the solutions a certain one is distinguished which has a particular form of growth at infinity.

The problem has been solved above of finding all self-similar solutions of the system (2) - (4) in the ξ,η plane in the sense of the accepted definition. As is readily understood, solutions that are self-similar in this sense will not, generally speaking, be self-similar solutions of the system (1) in the x, y plane in the analogous sense.

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Translated by M.D. Van D.